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LETTER TO THE EDITORS

A COMMENT ON "AN INTEGRAL EQUATION METHOD FOR DIFFUSION"

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NOMENCLATURE

T, temperature;
D, diffusion constant;
G, Green's function;
$g(t)$, $h(t)$, unknown source functions.

1. INTRODUCTION

In his article "An integral equation approach to diffusion", Shaw discussed the advantages of an integral equation for solving a diffusion problem numerically [1]. Integral equations have also proved to be a powerful technique for solving partial differential equations in various applications [2–10]. Starting with the thermal diffusion equation in a volume V with boundary S:

$$D\nabla^2 T = \frac{\partial T}{\partial t}. \tag{1}$$

Shaw wrote the solution as [1]:

$$T(\bar{r}', t') = \int_0^{t'} \mathrm{d}t \iint_S [G(\bar{r}|\bar{r}', t|t')\nabla T(\bar{r})\cdot \bar{u}_n - T(\bar{r})\nabla G(\bar{r}|\bar{r}', t|t')\cdot \bar{u}_n]\,\mathrm{d}S \tag{2}$$

where $G(\bar{r}|\bar{r}', t|t')$ is the Green's function of (1) and $\bar{u}_n$ the outward normal unity vector. $T(\bar{r}', 0)$ was taken zero for the sake of simplicity. By imposing the boundary conditions on the expression (2), one obtains an integral equation for the unknown function which can be either T or $\nabla T\cdot\bar{u}_n$ on the boundary S. If the temperature T is given on a part of the boundary, $\nabla T\cdot\bar{u}_n$ becomes then the unknown function and reversely, T will be the unknown function if $\nabla T\cdot\bar{u}_n$ is given on the remaining part of the boundary. Once the unknown function (T or $\nabla T\cdot\bar{u}_n$) has been determined by the integral equation, T can be calculated by (2) for arbitrary $\bar{r}'$ and t' values.

The expression (2) for the temperature T involves both the Green's function and its normal derivative. In writing a computer program for the integral equation one has to consider the fact that another kind of unknown appears when the type of boundary condition changes. In this paper it will be shown that the problem can be simplified by using an expression for T which only involves G. The method will be outlined for a simple thermal diffusion problem in a one-dimensional area.

2. INTEGRAL EQUATION

We consider a one-dimensional thermal diffusion problem described by the equation:

$$D\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \quad 0 \leqslant x \leqslant d \tag{3}$$

At $t = 0$, the temperature T is zero everywhere

$$T = 0 \quad \text{at} \quad t = 0. \tag{4}$$

For $t > 0$, the temperature at the left side ($x < 0$) is raised to T_1. Due to a convection mechanism, described by a thermal resistance, the boundary conditions at $x = 0$ and $x = d$ are:

$$D\frac{\partial T}{\partial x} = k(T - T_1) \quad \text{at} \quad x = 0 \tag{5}$$

$$T = 0 \quad \text{at} \quad x = d. \tag{6}$$

By using the Green's function of (3):

$$G(x,t) = \frac{1}{2\sqrt{(\pi D t)}}\exp\left(-\frac{x^2}{4Dt}\right) \tag{7}$$

the solution T is written as:

$$T(x,t) = \int_0^t g(t')G(x, t-t')\,\mathrm{d}t' + \int_0^t h(t')G(x-d, t-t')\,\mathrm{d}t' \tag{8}$$

where $g(t)$ and $h(t)$ are two unknown functions. Imposing the boundary conditions (5) and (6) on the proposed solution (8), one obtains:

$$\int_0^t g(t')\frac{\partial G(0, t-t')}{\partial x}\mathrm{d}t' + \int_0^t h(t')\frac{\partial G(-d, t-t')}{\partial x}\mathrm{d}t' = k\left[\int_0^t g(t')G(0, t-t')\,\mathrm{d}t' + \int_0^t h(t')G(-d, t-t')\,\mathrm{d}t' - T_1\right] \tag{9}$$

$$\int_0^t g(t')G(d, t-t')\,\mathrm{d}t' + \int_0^t h(t')G(0, t-t')\,\mathrm{d}t' = 0. \tag{10}$$

Equations (9) and (10) constitute two integral equations for the unknown functions $g(t)$ and $h(t)$. In contrast to Shaw [1], the relation (8) uses only the Green's function even when the boundary conditions involve T and $\partial T/\partial x$.

The integral equations (9) and (10) have been solved numerically in order to investigate the applicability of the procedure outlined here. For a detailed explanation of the numerical method one is referred to the literature [8]. In order to check the accuracy, the numerical results obtained for large values of t are compared with the analytical solution of (3) for $t \to \infty$:

$$T = \frac{kT_1}{1+k}\left(1 - \frac{x}{d}\right). \tag{11}$$

It was found that the accuracy was about 1% if the time increment Δt was less than $0.1d^2/D$. This result was found for a wide range of k values. The stability of the method was found to be extremely good. Even for high values of Δt ($= 0.5d^2/D$), the accuracy at large times was found to be 10%.

3. CONCLUSION

An integral equation method has been established for the thermal diffusion problem. The temperature was simply written as a superposition of Green's functions in order to construct an integral equation. By solving a particular example, the method has proved its validity.

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